Bootstrapping Empirical Processes

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Outline

The Bootstrap for Donsker Classes

The overall goal of this section is to prove the validity of the bootstrap central limit theorems given in Theorems 2.6 and 2.7 on Page 20 of Chapter 2.

Both unconditional and conditional multiplier central limit theorems play a pivotal role in this development.

An Unconditional Multiplier Central Limit Theorem

Theorem 10.1 (Multiplier central limit theorem)

Let $\mathcal{F}$ be a class of measurable functions, and let $\xi_1, ..., \xi_n$ be i.i.d. random variables with mean zero, variance 1, and with $\|\xi\|_{2,1} < \infty$, independent of the sample data $X_1, ..., X_n$. Let $\mathbb{G}_n' \equiv n^{-1/2} \sum_{i=1}^n \xi_i(\delta_{X_i} - P)$ and $\mathbb{G}_n'' \equiv n^{-1/2} \sum_{i=1}^n (\xi_i - \bar{\xi})\delta_{X_i}$, where $\bar{\xi} \equiv n^{-1} \sum_{i=1}^n \xi_i$. Then the following are equivalent:

(i) $\mathcal{F}$ is P-Donsker;
(ii) $\mathbb{G}_n'$ converges weakly to a tight process in $\ell^\infty(\mathcal{F})$;
(iii) $\mathbb{G}_n' \rightsquigarrow \mathbb{G}$ in $\ell^\infty(\mathcal{F})$;
(iv) $\mathbb{G}_n'' \rightsquigarrow \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

An Unconditional Multiplier Central Limit Theorem

Lemma 10.2 (Multiplier inequalities)

Let $Z_1, ..., Z_n$ be i.i.d. stochastic processes, with index $\mathcal{F}$ such that $E^*\|Z\|_{\mathcal{F}} < \infty$, independent of the i.i.d. Rademacher variables $\epsilon_1, ..., \epsilon_n$. Then for every i.i.d. sample $\epsilon_1, ..., \epsilon_n$ of real, mean-zero random variables independent of $Z_1, ..., Z_n$, and any $1 \leq n_0 \leq n$,

$$
\begin{aligned}
\frac{1}{2}\|\xi\|_1 E^*\Big\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\epsilon_i Z_i\Big\|_{\mathcal{F}} &\leq E^*\Big\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i Z_i\Big\|_{\mathcal{F}} \\
&\leq 2(n_0-1)E^*\|Z\|_{\mathcal{F}} E \max_{1\leq i\leq n}\frac{\|\xi_i\|}{\sqrt{n}} \\
&\quad + 2\sqrt{2}\|\xi\|_{2,1}\max_{n_0\leq k\leq n} E^*\Big\|\frac{1}{\sqrt{k}}\sum_{i=1}^{n}\epsilon_i Z_i\Big\|_{\mathcal{F}}.
\end{aligned}
$$

When the ξ_i are symmetrically distributed, the constants $1/2$, 2 and $2\sqrt{2}$ can all be replaced by 1.

Proof of Theorem 10.1

Proof

Note that the process $\mathbb{G}$, $\mathbb{G}_n$, $\mathbb{G}_n^{'}$ and $\mathbb{G}_n^{''}$ do not change if they are indexed by $\dot{\mathcal{F}} \equiv \{f - Pf : f \in \mathcal{F}\}$ rather than $\mathcal{F}$. Thus we can assume throughout the proof that $\|P\|_{\mathcal{F}} = 0$ without loss of generality.

(i) $\Leftrightarrow$ (ii): Convergence of the finite-dimensional marginal distributions of $\mathbb{G}_n$ and $\mathbb{G}_n^{'}$ is equivalent to $\mathcal{F} \subset L_2(P)$, and thus it suffices to show that the asymptotic equicontinuity conditions of both processes are equivalent. By Lemma 8.17, if $\mathcal{F}$ is Donsker, then $P^*(F > x) = o(x^{-2})$ as $x \to \infty$. Similarly, if $\xi \cdot \mathcal{F}$ is Donsker, then $P^*(|\xi| \times F > x) = o(x^{-2})$ as $x \to \infty$. In both cases, $P^*F < \infty$. Since the variance of ξ is finite, we have by Exercise 10.5.2 below that $E^* max_{1 \leq i \leq n} |\xi_i| / \sqrt{n} \to 0$. Combining this with Lemma 10.2, we have:

Proof of Theorem 10.1

Proof (cont).

$$\frac{1}{2}\|\xi\|_1 \limsup_{n\to\infty} E^*\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i f(X_i)\right\|_{\mathcal{F}_\delta} \le \limsup_{n\to\infty} E^*\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i f(X_i)\right\|_{\mathcal{F}_\delta}$$

$$\le 2\sqrt{2}\|\xi\|_{2,1} \sup_{k\ge n_0} E^*\left\|\frac{1}{\sqrt{k}}\sum_{i=1}^n \epsilon_i f(X_i)\right\|_{\mathcal{F}_\delta},$$

for every $\delta > 0$ and $n_0 \le n$. By the symmetrization theorem (Theorem 8.8), we can remove the Rademacher variables $\epsilon_1, ..., \epsilon_n$ at the cost of changing the constants.

Hence, for any sequence $\delta_n \downarrow 0$, $E^*\|n^{-1/2}\sum_{i=1}^n(\delta_{X_i} - P)\|_{\mathcal{F}_{\delta_n}} \to 0$ if and only if $E^*\|n^{-1/2}\sum_{i=1}^n \xi_i(\delta_{X_i} - P)\|_{\mathcal{F}_{\delta_n}} \to 0$. By Lemma 8.17, these mean versions of the asymptotic equicontinuity conditions imply the probability versions, and the desired results follow. We have actually proved that the first three assertions are equivalent.

Proof of Theorem 10.1

Proof (cont).
(iii)$\Rightarrow$ (iv): Note that by the equivalence of (i) and (iii), $\mathcal{F}$ is Glivenko-Cantelli. Since $\mathbb{G}_n' - \mathbb{G}_n'' = \sqrt{n}\bar{\xi}\mathbb{P}_n$, we now have that $\|\mathbb{G}_n' - \mathbb{G}_n''\|_{\mathcal{F}} \xrightarrow{P} 0$. Thus (iv) follows.

(iv)$\Rightarrow$(i): Let $(Y_1, ..., Y_n)$ be an independent copy of $(X_1, ..., X_n)$, and let $(\tilde{\xi}_1, ..., \tilde{\xi}_n)$ be an independent copy of $(\xi_1, ..., \xi_n)$, so that $(\xi_1, ..., \xi_n, \tilde{\xi}_1, ..., \tilde{\xi}_n)$ is independent of $(X_1, ..., X_n, Y_1, ..., Y_n)$. Let $\bar{\xi}$ be the pooled mean of the ξ_is and $\tilde{\xi}_i$s; set

$$\mathbb{G}_{2n}'' = (2n)^{-1/2}\left(\sum_{i=1}^{n}(\xi_i - \bar{\xi})\delta_{X_i} + \sum_{i=1}^{n}(\tilde{\xi}_i - \bar{\xi})\delta_{Y_i}\right)$$

and define

$$\tilde{\mathbb{G}}_{2n}'' = (2n)^{-1/2}\left(\sum_{i=1}^{n}(\tilde{\xi}_i - \bar{\xi})\delta_{X_i} + \sum_{i=1}^{n}(\xi_i - \bar{\xi})\delta_{Y_i}\right).$$

We now have that both $\mathbb{G}_{2n}'' \rightsquigarrow \mathbb{G}$ and $\tilde{\mathbb{G}}_{2n}'' \rightsquigarrow \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

Proof of Theorem 10.1

Proof (cont).

Thus, by the definition of weak convergence, we have that $(\mathcal{F}, \rho_P)$ is totally bounded and that for any sequence $\delta_n \downarrow 0$ both $\|\mathbb{G}''_{2n}\|_{\mathcal{F}_{\delta_n}} \xrightarrow{P} 0$ and $\|\tilde{\mathbb{G}}''_{2n}\|_{\mathcal{F}_{\delta_n}} \xrightarrow{P} 0$. Hence also $\|\mathbb{G}''_{2n} - \tilde{\mathbb{G}}''_{2n}\|_{\mathcal{F}_{\delta_n}} \xrightarrow{P} 0$. However, since

$$\mathbb{G}''_{2n} - \tilde{\mathbb{G}}''_{2n} = n^{-1/2} \sum_{i=1}^{n} \frac{(\xi_i - \tilde{\xi}_i)}{\sqrt{2}} (\delta_{X_i} - \delta_{Y_i}),$$

and since the weights $\check{\xi}_i \equiv (\xi_i - \tilde{\xi}_i)/\sqrt{2}$ satisfy the moment conditions for the theorem we are proving, we now have the $\check{\mathbb{G}}_n \equiv n^{-1/2} \sum_{i=1}^{n} (\delta_{X_i} - \delta_{Y_i}) \rightsquigarrow \sqrt{2}\mathbb{G}$ in $\ell^\infty(\mathcal{F})$ by the already proved equivalence between (iii) and (i). Thus, for any sequence $\delta_n \downarrow 0$, $E^*\|\check{\mathbb{G}}_n\|_{\mathcal{F}_{\delta_n}} \to 0$.

Proof of Theorem 10.1

Proof (cont).
Since also

$$E_Y\left|\sum_{i=1}^n f(X_i) - f(Y_i)\right| \geq \left|\sum_{i=1}^n f(X_i) - Ef(Y_i)\right| = \left|\sum_{i=1}^n f(X_i)\right|,$$

we can use Fubini's theorem to get

$$E^*\|\check{\mathbb{G}}_n\|_{\mathcal{F}_{\delta_n}} \geq E^*\|\mathbb{G}_n\|_{\mathcal{F}_{\delta_n}} \to 0.$$

Hence $\mathcal{F}$ is Donsker.

An Unconditional Multiplier Central Limit Theorem

Corollary 10.3 shows the possibly unexpected result that the multiplier empirical process is asymptotically independent of the usual empirical process, even though the same data $X_1, ..., X_n$ are used in both processes.

Corollary 10.3

Assume the conditions of Theorem 10.1 hold and that $\mathcal{F}$ is Donsker. Then $(\mathbb{G}_n, \mathbb{G}_n', \mathbb{G}_n'') \rightsquigarrow (\mathbb{G}, \mathbb{G}', \mathbb{G}'')$ in $[\ell^\infty(\mathcal{F})]^3$, where $\mathbb{G}$ and $\mathbb{G}'$ are independent P-Brownian bridges.

Proof of Corollary 10.3

Proof.

By the preceding theorem, the three processes are asymptotically tight marginally and hence asymptotically tight jointly. Since the first process is uncorrelated with the second process, the limiting distribution of the first process is independent of the limiting distribution of the second process. As argued in the proof of the multiplier central limit theorem, the uniform difference between $\mathbb{G}_n^{'}$ and $\mathbb{G}_n^{''}$ goes to zero in probability, and thus the remainder of the corollary follows.

Conditional Multiplier Central Limit Theorems

Theorem 10.4 (in-probability conditional multiplier central limit theorem)

Let $\mathcal{F}$ be a class of measurable functions, and let $\xi_1, \ldots, \xi_n$ be i.i.d. random variables with mean zero, variance 1, and with $\|\xi\|_{2,1} < \infty$, independent of the sample data $X_1, \ldots, X_n$. Let $\mathbb{G}'_n \equiv n^{-1/2} \sum_{i=1}^n \xi_i(\delta_{X_i} - P)$ and $\mathbb{G}''_n \equiv n^{-1/2} \sum_{i=1}^n (\xi_i - \bar{\xi})\delta_{X_i}$, where $\bar{\xi} \equiv n^{-1} \sum_{i=1}^n \xi_i$. Then the following are equivalent:

(i) $\mathcal{F}$ is Donsker;

(ii) $\mathbb{G}'_n \underset{\xi}{\overset{P}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$ and $\mathbb{G}'_n$is asymptotically measurable.

(iii) $\mathbb{G}''_n \underset{\xi}{\overset{P}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$ and $\mathbb{G}''_n$is asymptotically measurable.

Conditional Multiplier Central Limit Theorems

Lemma 10.5 is a conditional multiplier central limit theorem for i.i.d. Euclidean data.

Lemma 10.5

Let $Z_1, ..., Z_n$ be i.i.d. Euclidean random vectors, with $EZ = 0$ and $E\|Z\|^2 < \infty$, independent of the i.i.d. sequence of real random variables $\xi_1, ..., \xi_n$ with $E\xi = 0$ and $E\xi^2 = 1$. Then, conditionally on $Z_1, Z_2, ..., n^{-1/2}\sum_{i=1}^{n} \xi Z_i \rightsquigarrow N(0, covZ)$, for almost all sequences $Z_1, Z_2, ...$

Proof of Theorem 10.4

Proof

Since the process $\mathbb{G}$, $\mathbb{G}_n$, $\mathbb{G}_n'$ and $\mathbb{G}_n''$ are unaffected if the class $\mathcal{F}$ is replaced with $\{f - Pf : f \in \mathcal{F}\}$, we will assume $\|P\|_{\mathcal{F}} = 0$ throughout the proof, without loss of generality.

(i)⇒(ii): If $\mathcal{F}$ is Donsker, the sequence $\mathbb{G}_n'$ converges in distribution to a Brownian bridge process by the unconditional multiplier central limit theorem (Theorem 10.1). Thus $\mathbb{G}_n'$ is asymptotically measurable. By Lemma 8.17, a Donsker class is totally bounded by the semimetric $\rho_P(f, g) \equiv (P[f - g]^2)^{1/2}$. For each fixed $\delta > 0$ and $f \in \mathcal{F}$, denote $\Pi_\delta f$ to be the closest element in a given, finite δ-net (with respect to the metric ρ_P) for $\mathcal{F}$. We have by continuity of the limit process $\mathbb{G}$, that $\mathbb{G} \circ \Pi_\delta \to \mathbb{G}$, almost surely, as $\delta \downarrow 0$.

Proof of Theorem 10.4

Proof (cont).

Hence, for any sequence $\delta_n \downarrow 0$,

$$\sup_{h \in BL_1} |Eh(\mathbb{G} \circ \Pi_{\delta_n}) - Eh(\mathbb{G})| \to 0. \tag{10.2}$$

By Lemma 10.5, we also have for any fixed $\delta > 0$ that

$$\sup_{h \in BL_1} |E_\xi h(\mathbb{G}_n^{'} \circ \Pi_\delta) - Eh(\mathbb{G} \circ \Pi_\delta)| \to 0, \tag{10.3}$$

as $n \to \infty$, for almost all sequence $X_1, X_2, \ldots$. To prove (10.3), let $f_1, \ldots, f_m$ be the δ-mesh of $\mathcal{F}$ that defines Π_δ. Define the map $A : \mathbb{R}^m \mapsto \ell^\infty(\mathcal{F})$ by $(A(y))(f) = y_k$, where $y = (y_1, \ldots, y_m)$ and the integer k satisfies $\Pi_\delta f = f_k$. Now $h(\mathbb{G} \circ \Pi_\delta) = g(\mathbb{G}(f_1), \ldots, \mathbb{G}(f_m))$ for the function $g : \mathbb{R}^m \mapsto \mathbb{R}$ defined by $g(y) = h(A(y))$. It is not hard to see that h is bounded Lipschitz on $\ell^\infty(\mathcal{F})$, then g is also bounded Lipschitz on $\mathbb{R}^m$ with a Lipschitz norm no larger than the Lipschitz norm for h. So (10.3) follows from Lemma 10.5.

Proof of Theorem 10.4

Proof (cont).

Note that $BL_1(\mathbb{R}^m)$ is separable with respect to the metric $\rho_{(m)}(f,g) \equiv \sum_{i=1}^{\infty} 2^{-i} sup_{x \in K_i} |f(x) - g(x)|$, where $K_1 \subset K_2 \subset$ ... are compact sets satisfying $\cap_{i=1}^{\infty} K_i = \mathbb{R}^m$. Hence, since $\mathbb{G}' \circ \Pi_\delta$ and $\mathbb{G} \circ \Pi_\delta$ are both tight, the supremum in (10.3) can be replaced by a countable supremum. Thus the displayed quantity is measurable, since $h(\mathbb{G}' \circ \Pi_\delta)$ is measurable.

Now, still holding δ fixed,

$$\begin{aligned}
\sup_{h \in BL_1} |E_\xi h(\mathbb{G}_n' \circ \Pi_\delta) - E_\xi h(\mathbb{G}_n')| &\leq \sup_{h \in BL_1} E_\xi |h(\mathbb{G}_n' \circ \Pi_\delta) - h(\mathbb{G}_n')| \\
&\leq E_\xi \|\mathbb{G}_n' \circ \Pi_\delta - \mathbb{G}_n'\|_{\mathcal{F}}^* \\
&\leq E_\xi \|\mathbb{G}_n'\|_{\mathcal{F}_\delta}^*,
\end{aligned}$$

where $\mathcal{F}_\delta \equiv \{f - g : \rho_P(f,g) < \delta, f, g \in \mathcal{F}\}$.

Proof of Theorem 10.4

Proof (cont).

Thus the outer expectation of the left-hand-side is bounded above by $E^*\|\mathbb{G}_n^{'}\|_{\mathcal{F}_\delta}$. As we demonstrated in the proof of Theorem 10.1, $E^*\|\mathbb{G}_n^{'}\|_{\mathcal{F}_{\delta_n}} \to 0$, for any sequence $\delta_n \downarrow 0$. Now, we choose the sequence δ_n so that it goes to zero slowly enough to ensure that (10.3) still holds with δ replaced by δ_n. Combining this with (10.2), the desired result follows.

(ii)⇒(i): Let $h(\mathbb{G}_n^{'})^*$ and $h(\mathbb{G}_n^{'})_*$ denote measurable majorants and minorants with respect to $(\xi_1, ..., \xi_n, X_1, ..., X_n)$ jointly.

Proof of Theorem 10.4

Proof (cont).
By the triangle inequality and Fubini's theorem,

$$|E^* h(\mathbb{G}_n^{'}) - Eh(\mathbb{G})| \leq |E_X E_\xi h(\mathbb{G}_n^{'})^* - E_X^* E_\xi h(\mathbb{G}_n^{'})| \\ + E_X^* |E_\xi h(\mathbb{G}_n^{'}) - Eh(\mathbb{G})|,$$

where E_X denotes taking the expectation over $X_1, ..., X_n$. By (ii) and the dominated convergence theorem, the second term on the right side converges to zero for all $h \in BL_1$. Since the first term on the right is bounded above by $E_X E_\xi h(\mathbb{G}_n^{'})^* - E_X E_\xi h(\mathbb{G}_n^{'})_*$, it also converges to zero since $\mathbb{G}_n^{'}$ is asymptotically measurable. It is easy to see that the same result holds true if BL_1 is replaced by the class of all bounded, Lipschitz continuous nonnegative functions $h : \ell^\infty(\mathcal{F}) \mapsto \mathbb{R}$, and thus $\mathbb{G}_n^{'} \rightsquigarrow \mathbb{G}$ unconditionally by the Portmanteau theorem. Hence $\mathcal{F}$ is Donsker by the converse part of Theorem 10.1.

Proof of Theorem 10.4

Proof (cont).
(ii)⇒(iii): Since we can assume $\|P\|_{\mathcal{F}} = 0$, we have

$$|h(\mathbb{G}_n^{'}) - h(\mathbb{G}_n^{''})| \leq \|\bar{\xi}\mathbb{G}_n\|_{\mathcal{F}}. \tag{10.4}$$

Moreover, since (ii) also implies (i), we have that $E^*\|\bar{\xi}\mathbb{G}_n\|_{\mathcal{F}} \to 0$ by Lemma 8.17. Thus $sup_{h \in BL_1}|E_\xi h(\mathbb{G}_n^{'}) - E_\xi h(\mathbb{G}_n^{''})| \to 0$ in outer probability. Since (10.4) also implies that $\mathbb{G}^{''}$ is asymptotically measurable, (iii) follows.

(iii)⇒(i): As we did in the proof that (ii)⇒(i), it is not hard to show that $\mathbb{G}_n^{''} \rightsquigarrow \mathbb{G}$ unconditionally. So Theorem 10.1 gives us that $\mathcal{F}$ is Donsker.

Conditional Multiplier Central Limit Theorems

We now present the outer-almost-sure conditional multiplier central limit theorem:

Theorem 10.6

Assume the conditions of Theorem 10.4. Then the following are equivalent:

(i) $\mathcal{F}$ is Donsker and $P^*\|f - Pf\|_{\mathcal{F}}^2 < \infty$;

(ii) $\mathbb{G}_n' \underset{\xi}{\overset{as*}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

(iii) $\mathbb{G}_n'' \underset{\xi}{\overset{as*}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

Proof of Theorem 10.6

Proof.

The equivalence of (i) and (ii) is given in Theorem 2.9.7 of van der Vaart and Wellner (2000).

(ii)⇒(iii): As in the proof of Theorem 10.4, we assume that $\|P\|_{\mathcal{F}} = 0$ throughout the proof, without loss of generality. Since

$$|h(\mathbb{G}_n^{'}) - h(\mathbb{G}_n^{''})| \leq |\sqrt{n}\bar{\xi}| \times \|\mathbb{P}_n\|_{\mathcal{F}}, \tag{10.5}$$

for any $h \in BL_1$, we have

$$\sup_{h \in BL_1} |E_\xi h(\mathbb{G}_n^{'}) - E_\xi h(\mathbb{G}_n^{''})| \leq E_\xi |\sqrt{n}\bar{\xi}| \times \|\mathbb{P}_n\|_{\mathcal{F}} \leq \|\mathbb{P}_n\|_{\mathcal{F}} \overset{as*}{\to} 0,$$

since the equivalence of (i) and (ii) implies that $\mathcal{F}$ is both Donsker and Glivenko-Cantelli.

Proof of Theorem 10.6

Proof (cont).
Hence,

$$\sup_{h \in BL_1} |E_\xi h(\mathbb{G}_n^{''}) - Eh(\mathbb{G})| \overset{as*}{\to} 0.$$

(10.5) also yields that $E_\xi h(\mathbb{G}_n^{''})^* - E_\xi h(\mathbb{G}_n^{''})_* \overset{as*}{\to} 0$, and thus (iii) follows.

(iii)⇒(ii): Let $h \in BL_1$. Since $E_\xi h(\mathbb{G}_n^{''})^* - Eh(\mathbb{G}) \overset{as*}{\to} 0$, we have $E^* h(\mathbb{G}_n^{''}) \to Eh(\mathbb{G})$. Since this holds for all $h \in BL_1$, we now have that $\mathbb{G}_n^{''} \rightsquigarrow \mathbb{G}$ unconditionally by the Portmanteau theorem. Then by Theorem 10.1, $\mathcal{F}$ is both Donsker and Glivenko-Cantelli. Thus from (10.5), we get (ii).

Bootstrap Central Limit Theorems

Recall $\hat{\mathbb{P}}_n \equiv n^{-1}\sum_{i=1}^{n} W_{ni}\delta_{X_i}$ and $\hat{\mathbb{G}}_n \equiv \sqrt{n}(\hat{\mathbb{P}}_n - \mathbb{P}_n)$.
$\tilde{\mathbb{P}}_n \equiv n^{-1}\sum_{i=1}^{n}(\xi/\bar{\xi})\delta_{X_i}$ and $\tilde{\mathbb{G}}_n \equiv \sqrt{n}(\mu/\tau)(\tilde{\mathbb{P}}_n - \mathbb{P}_n)$, where the weights $\xi_1, ..., \xi_n$ are i.i.d. nonnegative, independent of $X_1, ..., X_n$, with mean $0 < \mu < \infty$ and variance $0 < \tau^2 < \infty$, and with $\|\xi\|_{2,1} < \infty$

Theorem 2.6

The following are equivalent:

(i) $\mathcal{F}$ is P-Donsker.

(ii) $\hat{\mathbb{G}}_n \underset{W}{\overset{P}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$ and the sequence $\hat{\mathbb{G}}_n$ is asymptotically measurable.

(iii) $\tilde{\mathbb{G}}_n \underset{\xi}{\overset{P}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$ and the sequence $\tilde{\mathbb{G}}_n$ is asymptotically measurable.

Proof of Theorem 2.6

Proof.

The equivalence of (i) and (ii) follows from Theorem 3.6.1 of van der Vaart and Wellner (2000).

(i)⇔(iii): Let $\xi_i^0 \equiv \tau^{-1}(\xi_i - \mu)$, $i = 1, ..., n$, and define $\mathbb{G}_n^0 \equiv n^{-1/2} \sum_{i=1}^n (\xi_i^0 - \bar{\xi^0})\delta_{X_i}$, where $\bar{\xi^0} \equiv n^{-1} \sum_{i=1}^n \xi_i^0$. The basic idea is to show the asymptotic equivalence of $\tilde{\mathbb{G}}_n$ and $\mathbb{G}_n^0$. Then we can use Theorem 10.4 to get the result.

From the definition we can get that

$$\mathbb{G}_n^0 - \tilde{\mathbb{G}}_n = \left(1 - \frac{\mu}{\bar{\xi}}\right) \mathbb{G}_n^0 = \left(\frac{\bar{\xi}}{\mu} - 1\right) \tilde{\mathbb{G}}_n. \tag{10.6}$$

Proof of Theorem 2.6

Proof (cont).
(i)⇒(iii): Since $\xi_1^0, ..., \xi_n^0$ satisfy the conditions of the unconditional multiplier central limit theorem, we have that $\mathbb{G}_n^0 \rightsquigarrow \mathbb{G}$. Theorem 10.4 also implies that $\mathbb{G}_n^0 \underset{\xi}{\overset{P}{\rightsquigarrow}} \mathbb{G}$. From (10.6) we can get $\|\tilde{\mathbb{G}}_n - \mathbb{G}_n^0\|_{\mathcal{F}} \xrightarrow{P} 0$, so $\tilde{\mathbb{G}}_n$ is asymptotically measurable and

$$\sup_{h \in BL_1} |E_\xi h(\mathbb{G}_n^0) - E_\xi h(\tilde{\mathbb{G}}_n)| \xrightarrow{P} 0.$$

(iii)⇒(i): Just as the proof of Theorem 10.4, we can get that $\tilde{\mathbb{G}}_n \rightsquigarrow \mathbb{G}$ in $\ell^\infty(\mathcal{F})$ unconditionally. The unconditional multiplier central limit theorem now verifies that $\mathcal{F}$ is Donsker, and thus we get the result.

Bootstrap Central Limit Theorems

Theorem 2.7

The following are equivalent:

(i) $\mathcal{F}$ is P-Donsker and $P^*[sup_{f\in\mathcal{F}}(f(X)-Pf)^2] < \infty$.

(ii) $\hat{\mathbb{G}}_n \underset{W}{\overset{as*}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

(iii) $\tilde{\mathbb{G}}_n \underset{\xi}{\overset{as*}{\rightsquigarrow}} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.