Functional Delta Method

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1. §12.1 Proofs for main theorems related to functional delta method

2. §12.2 Examples of Hadamard differentiable maps

From previous talk

Hadamard-differentiable

A map $\phi : \mathbb{D}_\phi \mapsto \mathbb{E}$ is Hadamard-differentiable at $\theta \in \mathbb{D}$, tangentially to a set $\mathbb{D}_0 \subset \mathbb{D}$, if there exists a continuous linear map $\phi'_\theta : \mathbb{D} \mapsto \mathbb{E}$ such that

$$\frac{\phi\left(\theta + t_n h_n\right) - \phi(\theta)}{t_n} \to \phi'_\theta(h)$$

as $n \to \infty$, for all converging sequences $t_n \to 0$ and $h_n \to h \in \mathbb{D}_0$, with $h_n \in \mathbb{D}$ and $\theta + t_n h_n \in \mathbb{D}_\phi$ for all $n \geq 1$ sufficiently large.

THEOREM 2.8 (functional delta method theorem)

For normed spaces $\mathbb{D}$ and $\mathbb{E}$, let $\phi : \mathbb{D}_\phi \subset \mathbb{D} \mapsto \mathbb{E}$ be Hadamard-differentiable at θ tangentially to $\mathbb{D}_0 \subset \mathbb{D}$. Assume that $r_n (X_n - \theta) \rightsquigarrow X$ for some sequence of constants $r_n \to \infty$, where X_n takes its values in $\mathbb{D}_\phi$, and X is a tight process taking its values in $\mathbb{D}_0$. Then $r_n (\phi (X_n) - \phi(\theta)) \rightsquigarrow \phi'_\theta(X)$

Proofs

Consider the map $h \mapsto r_n(\phi(\theta + r_n^{-1} h) - \phi(\theta)) \equiv g_n(h)$, defined on the domain $\mathbb{D}_n \equiv \left\{h : \theta + r_n^{-1} h \in \mathbb{D}_\phi\right\}$ and satisfies $g_n (h_n) \to \phi'_\theta(h)$ for every $h_n \to h \in \mathbb{D}_0$ with $h_n \in \mathbb{D}_n$.

By extended continuous mapping theorem, we have

$$g_n(r_n(X_n - \theta)) = r_n(\phi(X_n) - \phi(\theta)) \to \phi'_\theta(X)$$

It suffices that ϕ is differentiable at just one single point θ, which is convenient in empirical process where random elements are abstract and continuous differentiability may fail.

$\mathbb{X}_n\,(X_n)$ is a sequence of random elements in a normed space $\mathbb{D}$ based on the data sequence $\{X_n, n \geq 1\}$, while $\hat{\mathbb{X}}_n\,(X_n, W_n)$ is a bootstrapped version of $\mathbb{X}_n$ based on both the data sequence and a sequence of weights $W = \{W_n, n \geq 1\}$.

THEOREM 12.1 Delta method bootstrap

For normed spaces $\mathbb{D}$ and $\mathbb{E}$, let $\phi : \mathbb{D}_\phi \subset \mathbb{D} \mapsto \mathbb{E}$ be Hadamard-differentiable at μ tangentially to $\mathbb{D}_0 \subset \mathbb{D}$, with derivative ϕ'_μ. Let $\mathbb{X}_n$ and $\hat{\mathbb{X}}_n$ have values in $\mathbb{D}_\phi$, with $r_n\,(\mathbb{X}_n - \mu) \rightsquigarrow \mathbb{X}$, where $\mathbb{X}$ is tight and takes its values in $\mathbb{D}_0$ for some sequence of constants $0 < r_n \to \infty$, the maps $W_n \mapsto h\left(\hat{\mathbb{X}}_n\right)$ are measurable for every $h \in C_b(\mathbb{D})$ outer almost surely, and where $r_n c\left(\hat{\mathbb{X}}_n - \mathbb{X}_n\right) \underset{\mathsf{W}}{\overset{\mathsf{P}}{\rightsquigarrow}} \mathbb{X}$, for a constant $0 < c < \infty$. Then $r_n c\left(\phi\left(\hat{\mathbb{X}}_n\right) - \phi\left(\mathbb{X}_n\right)\right) \underset{\mathsf{W}}{\overset{\mathsf{P}}{\rightsquigarrow}} \phi'_\mu(\mathbb{X})$

The sequence of "conditional random laws" (given $X_1, X_2, \ldots$) of $\sqrt{n}\left(\phi\left(\hat{\mathbb{X}}_n\right) - \phi\left(\mathbb{X}_n\right)\right)$ is asymptotically consistent in probability for estimating the laws of the random elements $\sqrt{n}\left(\phi\left(\mathbb{X}_n\right) - \phi(\mu)\right)$

Proofs for bootstrap delta method

We can show that $\hat{\mathbb{U}}_n \equiv r_n\left(\hat{\mathbb{X}}_n - \mathbb{X}_n\right) \rightsquigarrow c^{-1}\mathbb{X}$ and $r_n\left(\hat{\mathbb{X}}_n - \mu\right) \rightsquigarrow Z$ unconditionally, where Z is a tight random element. Using the same strategy as proving the conditional multiplier central limit theorem.

Fix some $h \in BL_1(\mathbb{D})$, define $\mathbb{U}_n \equiv r_n\left(\mathbb{X}_n - \mu\right)$ and let $\tilde{\mathbb{X}}_1$ and $\tilde{\mathbb{X}}_2$ be two independent copies of $\mathbb{X}$, we have

$$
\begin{aligned}
&\left| \mathrm{E}^* h\left(\hat{\mathbb{U}}_n + \mathbb{U}_n\right) - \mathrm{E} h\left(c^{-1}\tilde{\mathbb{X}}_1 + \tilde{\mathbb{X}}_2\right) \right| \\
&\leq \left| \mathrm{E}_{X_n}\mathrm{E}_{W_n} h\left(\hat{\mathbb{U}}_n + \mathbb{U}_n\right)^* - \mathrm{E}^*\mathrm{E}_{W_n} h\left(\hat{\mathbb{U}}_n + \mathbb{U}_n\right)\right| \to 0 \\
&\quad + \mathrm{E}^*\left|\mathrm{E}_{W_n} h\left(\hat{\mathbb{U}}_n + \mathbb{U}_n\right) - \mathrm{E}_{\tilde{\mathbb{X}}_1} h\left(c^{-1}\tilde{\mathbb{X}}_1 + \mathbb{U}_n\right)\right| \to 0 \\
&\quad + \left|\mathrm{E}^*\mathrm{E}_{\tilde{\mathbf{X}}_1} h\left(c^{-1}\tilde{\mathbb{X}}_1 + \mathbb{U}_n\right) - \mathrm{E}_{\tilde{\mathbf{X}}_2}\mathrm{E}_{\tilde{\mathbf{X}}_1} h\left(c^{-1}\tilde{\mathbb{X}}_1 + \tilde{\mathbb{X}}_2\right)\right| \to 0
\end{aligned}
$$

(Asymptotic measureability); $(\hat{\mathbb{U}}_n \equiv r_n\left(\hat{\mathbb{X}}_n - \mathbb{X}_n\right) \rightsquigarrow c^{-1}\mathbb{X}, h \in BL_1(\mathbb{D}))$;
$(\mathbb{U}_n \rightsquigarrow \mathbb{X},\ h \in BL_1(\mathbb{D}))$

(Proof Con't)

h is arbituary, and by portmanteau theorem we have that unconditionally,

$$r_n \begin{pmatrix} \hat{\mathbb{X}}_n - \mu \\ \mathbb{X}_n - \mu \end{pmatrix} \rightsquigarrow \begin{pmatrix} c^{-1}\tilde{\mathbb{X}}_1 + \tilde{\mathbb{X}}_2 \\ \tilde{\mathbb{X}}_2 \end{pmatrix}$$

Then we apply the functional delta method

$$r_n \begin{pmatrix} \phi\left(\hat{\mathbb{X}}_n\right) - \phi(\mu) \\ \phi\left(\mathbb{X}_n\right) - \phi(\mu) \\ \hat{\mathbb{X}}_n - \mu \\ \mathbb{X}_n - \mu \end{pmatrix} \rightsquigarrow \begin{pmatrix} \phi'_\mu\left(c^{-1}\tilde{\mathbb{X}}_1 + \mathbb{X}_2\right) \\ \phi'_\mu\left(\mathbb{X}_2\right) \\ c^{-1}\tilde{\mathbb{X}}_1 + \mathbb{X}_2 \\ \mathbb{X}_2 \end{pmatrix}$$

Which implies

$$r_n c \begin{pmatrix} \phi\left(\hat{\mathbb{X}}_n\right) - \phi\left(\mathbb{X}_n\right) \\ \hat{\mathbb{X}}_n - \mathbb{X}_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} \phi'_\mu(\mathbb{X}) \\ \mathbb{X} \end{pmatrix}$$

since ϕ'_μ is linear on $\mathbb{D}_0$

(Proof Con't)

With the map $(x, y) \mapsto x - \phi'_\mu(y)$, we have unconditionally

$$r_n c\left(\phi\left(\hat{\mathbb{X}}_n\right) - \phi\left(\mathbb{X}_n\right)\right) - \phi'_\mu\left(r_n c\left(\hat{\mathbb{X}}_n - \mathbb{X}_n\right)\right) \quad \xrightarrow{\mathrm{P}} \quad 0$$

By bootstrap continuous mapping theorem,

$$\phi'_\mu\left(r_n c\left(\hat{\mathbb{X}}_n - \mathbb{X}_n\right)\right) \underset{W}{\overset{\mathrm{P}}{\rightsquigarrow}} \phi'_\mu(\mathbb{X})$$

Thus completes the proof

1. §12.1 Proofs for main theorems related to functional delta method

2. §12.2 Examples of Hadamard differentiable maps

Composition

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a fixed map. Given an arbitrary set $\mathcal{X}$, consider the map $\phi : \ell^\infty(\mathcal{X}) \mapsto \ell^\infty(\mathcal{X})$ given by $\phi(A)(x) = g(A(x))$.
One simple example is $\phi(f) = 1/f$, In §2.2.4, we established that ϕ is Hadamard-differentiable with derivative at θ equal to $h \mapsto -h/\theta^2$

Lemma 12.2 Hadamard-differentiable of composition mapping

Let $g : B \subset \mathbb{R} \mapsto \mathbb{R}$ be differentiable with derivative continuous on all closed subsets of B, and let $\mathbb{D}_\phi = \{A \in \ell^\infty(\mathcal{X}) : \{R(A)\}^\delta \subset B \text{ for some } \delta > 0\}$, where $\mathcal{X}$ is a set, $R(C)$ denotes the range of the function $C \in \ell^\infty(\mathcal{X})$, and D^δ is the δ-enlargement of the set D.
Then $A \mapsto g \circ A$ is Hadamard-differentiable as a map from $\mathbb{D}_\phi \subset \ell^\infty(\mathcal{X})$ to $\ell^\infty(\mathcal{X})$, at every $A \in \mathbb{D}_\phi$.
The derivative is given by $\phi'_A(\alpha) = g'(A)\alpha$, where g' is the derivative of g.

Proof of Lemma 12.2.

Let t_n be any real sequence with $t_n \to 0$, let $\{h_n\} \in \ell^\infty(\mathcal{X})$ be any sequence converging to $h \in \ell^\infty(\mathcal{X})$ uniformly, and define $A_n = A + t_n h_n$. Then, by the conditions of the theorem, there exists a closed $B_1 \subset B$ such that $\{R(A) \cup R(A_n)\}^\delta \subset B_1$ for some $\delta > 0$ and all n large enough. Hence

$$\sup_{x \in \mathcal{X}} \left| \frac{g(A(x) + t_n h_n(x)) - g(A(x))}{t_n} - g'(A(x)) h(x) \right| \to 0$$

as $n \to \infty$, since continuous functions on closed sets are bounded and thus g' is uniformly continuous on B_1.$\square$

Integration

$$\phi(A, B) = \int_{(a,b]} A(s)\,dB(s) \quad \text{and} \quad \psi(A, B)(t) = \int_{(a,t]} A(s)\,dB(s)$$

are Hadamard differentiable. Where $A \in D[a, b]$ is a cadlag function on the interval $[a, b] \subset \overline{\mathbb{R}}$ and $B \in BV_M[a, b]$ is a cadlag function with total variation bounded by a fixed constant M

Lemma 12.3

Let $\mathbb{D}_M \equiv D[a, b] \times BV_M[a, b]$. For each fixed $M < \infty$, the maps $\phi : \mathbb{D}_M \mapsto \mathbb{R}$ and $\psi : \mathbb{D}_M \mapsto D[a, b]$ defined previously are Hadamard differentiable at each $(A, B) \in \mathbb{D}_M$ with $\int_{(a,b]} |dA| < \infty$..
The derivatives are given by

$$\phi'_{A,B}(\alpha, \beta) = \int_{(a,b]} A d\beta + \int_{(a,b]} \alpha dB, \quad \text{and}$$

$$\psi'_{A,B}(\alpha, \beta)(t) = \int_{(a,t]} A d\beta + \int_{(a,t]} \alpha dB$$

Note that if β is not of bounded variation then $\int A d\beta$ can be defined via integration by parts.

Example (Wilcoxon statistics)

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from distributions F and G on the reals. If $\mathbb{F}_m$ and $\mathbb{G}_n$ are the respective empirical distribution functions. Then $T = \int_{\mathbb{R}} \mathbb{G}_n(x) d\mathbb{F}_m(x)$ is the Mann-Whitney form of the Wilcoxon statistics, and it's an estimator of $\phi(G, F) = \int GdF = \mathrm{P}(Y \leq X)$
Note that $F, G, \mathbb{F}_m$ and $\mathbb{G}_n$ all have total variation ≤ 1. Lemma 12.3 gives
$\phi'_{G,F}(\alpha, \beta) = \int_{\mathbb{R}} Gd\beta + \int_{\mathbb{R}} \alpha dF$
If we assume $m/(m+n) \to \lambda \in [0, 1]$ as $m \wedge n \to \infty$,

$$\sqrt{\frac{mn}{m+n}} \begin{pmatrix} \mathbb{G}_n - G \\ \mathbb{F}_m - F \end{pmatrix} \rightsquigarrow \begin{pmatrix} \sqrt{\lambda}\mathbb{G}_G \\ \sqrt{1-\lambda}\mathbb{G}_F \end{pmatrix}$$

where $\mathbb{G}_F, \mathbb{G}_G$ are independent tight, F- and G-Brownian bridge processes.

$$\sqrt{\frac{mn}{m+n}} \left(\int_{\mathbb{R}} \mathbb{G}_n(x) d\mathbb{F}_m(x) - \int GdF \right) \rightsquigarrow \sqrt{\lambda} \int_{\mathbb{R}} \mathbb{G}_G dF + \sqrt{1-\lambda} \int_{\mathbb{R}} Gd\mathbb{G}_F$$

Example (Nelson-Aalen)

$X = T \wedge C$ is the minimum of a failure time T and censoring time C, and $\delta = 1\{T \le C\}$. T and C are independent. $\Lambda(t) = \int_0^t dF(s)/S(s-)$
The Nelson-Aalen estimator for Λ based on samples is

$$\hat{\Lambda}_n(t) \equiv \int_{[0,t]} \frac{d\hat{N}_n(s)}{\hat{Y}_n(s)}$$

$$\hat{N}_n(t) \equiv n^{-1} \sum_{i=1}^{n} \delta_i 1\left\{X_i \le t\right\} \text{ and } \hat{Y}_n(t) \equiv n^{-1} \sum_{i=1}^{n} 1\left\{X_i \ge t\right\}$$

Example (Nelson-Aalen Con't)

$\{\delta 1\{X \leq t\}, t \geq 0\}$ and $\{1\{X \geq t\} : t \geq 0\}$ can be verified to be both Donsker class, therefore

$$\sqrt{n}\begin{pmatrix} \hat{N}_n - N_0 \\ \hat{Y}_n - Y_0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{G}_1 \\ \mathbb{G}_2 \end{pmatrix}$$

where $N_0(t) \equiv P(T \leq t, C \geq T)$, $Y_0(t) \equiv P(X \geq t)$. $\mathbb{G}_1$ and $\mathbb{G}_2$ are tight Gaussian processes.

The N-A estimator depends on the pairs $\left(\hat{N}_n, \hat{Y}_n\right)$ through

$$(A, B) \mapsto \left(A, \frac{1}{B}\right) \mapsto \int_{[0,t]} \frac{1}{B} dA$$

In this case, the point (A, B) of interest is $A = N_0$ and $B = Y_0$.

Example (Nelson-Aalen Con't)

By Lemma 12.3 and chain rule of Hadamard differentiability, we can see such composition map is Hadamard differentiable on a domain of type $\{(A, B) : \int_{[0,\tau]} |dA(t)| \leq M, \inf_{t\in[0,\tau]} |B(t)| \geq \epsilon\}$ for $M < 0$ and $\epsilon > 0$ at every point (A, B) such that $1/B$ is of bounded variation.
The derivative of such composition map is given by

$$(\alpha, \beta) \mapsto \left(\alpha, \frac{-\beta}{Y_0^2}\right) \mapsto \int_{[0,t]} \frac{d\alpha}{Y_0} - \int_{[0,t]} \frac{\beta\, dN_0}{Y_0^2}$$

By functional delta method

$$\sqrt{n}\left(\hat{\Lambda}_n - \Lambda\right) \rightsquigarrow \int_{[0,(\cdot)]} \frac{d\mathbb{G}_1}{Y_0} - \int_{[0,(\cdot)]} \frac{\mathbb{G}_2\, dN_0}{Y_0^2}$$

Example (Nelson-Aalen Con't)

$$\sqrt{n}\left(\hat{\Lambda}_n - \Lambda\right) \rightsquigarrow \int_{[0,(\cdot)]} \frac{d\mathbb{G}_1}{Y_0} - \int_{[0,(\cdot)]} \frac{\mathbb{G}_2 \, dN_0}{Y_0^2}$$

The right side is equal to $\int_{[0,(\cdot)]} d\mathbb{M}/Y_0$, where $\mathbb{M}(t) \equiv \mathbb{G}_1(t) - \int_{[0,t]} \mathbb{G}_2 \, d\Lambda$
$\mathbb{M}(t)$ can be shown to be a Gaussian martingale with independent increments.

Proof for Lemma 12.3

Recall Lemma 12.3

$$\phi(A, B) = \int_{(a,b]} A(s)\,dB(s) \quad \text{and} \quad \psi(A, B)(t) = \int_{(a,t]} A(s)\,dB(s)$$

are Hadamard differentiable at each $(A, B) \in \mathbb{D}_M \equiv D[a, b] \times BV_M[a, b]$
And the derivatives are given by

$$\phi'_{A,B}(\alpha, \beta) = \int_{(a,b]} A\,d\beta + \int_{(a,b]} \alpha\,dB, \quad \text{and}$$

$$\psi'_{A,B}(\alpha, \beta)(t) = \int_{(a,t]} A\,d\beta + \int_{(a,t]} \alpha\,dB$$

Proof of Lemma 12.3

For sequences $t_n \to 0, \alpha_n \to \alpha,$ and $\beta_n \to \beta$, define $A_n \equiv A + t_n\alpha_n$ and $B_n \equiv B + t_n\beta_n$. Require that $(A_n, B_n) \in \mathbb{D}_M$, therefore the total variation of B_n is boudned by M

$$
\begin{aligned}
&\frac{\int_{(a,t]} A_n dB_n - \int_{(a,t]} A dB}{t_n} - \psi'_{A,B}(\alpha_n, \beta_n) \\
=&\frac{\int_{(a,t]}(A + t_n\alpha_n) dB_n - \int_{(a,t]} A dB}{t_n} - \left(\int_{(a,t]} A d\beta_n + \int_{(a,t]} \alpha_n dB\right) \\
=&\int_{(a,t]} \alpha d(B_n - B) + \int_{(a,t]} (\alpha_n - \alpha) d(B_n - B)
\end{aligned}
$$

Next show the right side goes to 0. First note that the second term goes to zero uniformly over $t \in (a, b]$ since both B_n and B have total variation boudned by M and $\alpha_n \to \alpha$

Proof of Lemma 12.3 Con't

In terms of the first term $\int_{(a,t]} \alpha d(B_n - B)$.
For a fixed $\epsilon > 0$, since α is cadlag, there exists a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that α varies less than ϵ over each interval $[t_{i-1}, t_i), 1 \leq i \leq m,$ and $m < \infty$.
Define $\tilde{\alpha} = \alpha(t_{i-1})$ over the interval $[t_{i-1}, t_i)$ and $\tilde{\alpha}(b) = \alpha(b)$

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Proof of Lemma 12.3 Con't

$$
\begin{aligned}
& \left\| \int_{(a,t]} \alpha\, d\left(B_n - B\right) \right\|_\infty \\
\leq & \left\| \int_{(a,t]} (\alpha - \tilde{\alpha})\, d\left(B_n - B\right) \right\|_\infty + \left\| \int_{(a,t]} \tilde{\alpha}\, d\left(B_n - B\right) \right\|_\infty \\
& \leq \quad \|\alpha - \tilde{\alpha}\|_\infty 2M + \sum_{i=1}^{m} \left|\alpha\left(t_{i-1}\right)\right| \times \left|\left(B_n - B\right)\left(t_i\right) - \left(B_n - B\right)\left(t_{i-1}\right)\right| \\
& + |\alpha(b)| \times \left|\left(B_n - B\right)(b)\right| \\
& \leq \quad \epsilon 2M + (2m+1) \left\|B_n - B\right\|_\infty \|\alpha\|_\infty \\
\to & \epsilon 2M
\end{aligned}
$$

Thus we have proved

$$
\frac{\int_{(a,t]} A_n dB_n - \int_{(a,t]} A dB}{t_n} \to \psi'_{A,B}(\alpha, \beta)
$$

and the desired Hadamard differentiability of ψ follows

The product integral

For a function $A \in D(0, b]$, let $\Delta A(t) = A(t) - A(t-)$ and $A^c(t) \equiv A(t) - \sum_{0<s\leq t} \Delta A(s)$ be the jump part and continuous part of A, respectively.

Product integral

The product integral is defined as

$$\phi(A)(t) \equiv \prod_{0<s\leq t} (1 + dA(s)) = \prod_{0<s\leq t} (1 + \Delta A(s)) \exp\left(A^c(t)\right)$$

The expression in the middle is simply a notation, as motivated by the mathematical definition of the product integral.
We will also use the notation

$$\phi(A)(s, t] = \prod_{s<u\leq t} (1 + dA(u)) \equiv \frac{\phi(A)(t)}{\phi(A)(s)}$$

for all $0 \leq s < t$

Product integration Hadamard differentiability

Lemma 12.5

For fixed constants $0 < b, M < \infty$, the product integral map $\phi : BV_M[0, b] \subset D[0, b] \mapsto D[0, b]$ is Hadamard differentiable with derivative

$$\phi'_A(\alpha)(t) = \int_{(0,t]} \phi(A)(0, u)\phi(A)(u, t]\, d\alpha(u)$$

When $\alpha \in D[0, b]$ has unbounded variation, the above quantity is welldefined by integration by parts.

Application

It can be shown that in the right-censored survival setting $S(t) = \phi(-\Lambda)(t)$, and that the Kaplan-Meier estimator $\hat{S}_n(t) = \phi\left(-\hat{\Lambda}_n\right)(t)$.

Apply Lemma 12.5, we can derive the asymptotic limit distribution of $\sqrt{n}\left(\hat{S}_n - S\right)$

Recall from slide 20, the asymtotic limit distribution of $\sqrt{n}\left(\hat{\Lambda}_n - \Lambda\right)$ in $D[0,\tau]$ is

$$\sqrt{n}\left(\hat{\Lambda}_n - \Lambda\right) \rightsquigarrow \int_{[0,(\cdot)]} d\mathbb{M}/Y_0 = \mathbb{Z}$$

If the N-A estimator $\hat{\Lambda}_n$ is of uniformly bounded total variation (with probability tending to 1), then delta-method gives

$$\begin{aligned}\sqrt{n}\left(\hat{S}_n - S\right) \rightsquigarrow \phi'_{-\Lambda}(-\mathbb{Z}) &= -\int_{(0,(\cdot)]} \phi(-\Lambda)(0,u)\phi(-\Lambda)(u,t]\, d\mathbb{Z} \\ &= -S(t)\int_{(0,(\cdot)]} \frac{d\mathbb{Z}}{(1-\Delta\Lambda)}\end{aligned}$$

Inversion

For a non-decreasing function $B \in D(-\infty, \infty)$, define the left-continuous inverse

$$r \mapsto B^{-1}(r) \equiv \inf\{x : B(x) \geq r\}$$

We will hereafter use the notation $\tilde{D}[u, v]$ to denote all left-continuous functions with right-hand limits (caglad) on $[u, v]$ and $D_1[u, v]$ to denote the restriction of all non-decreasing functions in $D(-\infty, \infty)$ to the interval $[u, v]$

Lemma 12.7

Let $-\infty < p \leq q < \infty$, and let the non-decreasing function $A \in D(-\infty, \infty)$ be continuously differentiable on the interval

$$[u, v] \equiv \left[A^{-1}(p) - \epsilon, A^{-1}(q) + \epsilon\right]$$

for some $\epsilon > 0$, with derivative A' being strictly positive and bounded over $[u, v]$ Then the inverse map $B \mapsto B^{-1}$ as a map

$$D_1[u, v] \subset D[u, v] \mapsto \tilde{D}[p, q]$$

is Hadamard differentiable at A tangentially to $C[u, v]$, with derivative

$$\alpha \mapsto -\left(\alpha / A'\right) \circ A^{-1}$$

There are also results similar to the one above but utilizes the knowledge about the support of the distribution function F

Application

An important application of these results is estimation and inference for the quantile function $p \mapsto F^{-1}(p)$ based on the empirical distribution function for i.i.d data.
These results are applicable to other estimators of the distribution function F besides the usual empirical distribution, provided the standardized estimators converge to a tight limiting process over the necessary intervals.
We now apply Lemma 12.7 to the construction of quantile processes based on the Kaplan-Meier estimator described above.

Application to Kaplan-Meier quantile process

The Kaplan-Meier quantile process is defined as

$$\left\{\hat{\xi}(p) \equiv \hat{F}_n^{-1}(p), 0 < p \leq q\right\}$$

Assume that F is continously differentiable on $[0, \tau]$ with f bounded below by zero and finite.

Combine with the results that

$$\sqrt{n}\left(\hat{S}_n - S\right) \rightsquigarrow -S(t) \int_{(0,(\cdot)]} \frac{d\mathbb{Z}}{(1 - \Delta\Lambda)}$$

$$\sqrt{n}(\hat{\xi} - \xi)(\cdot) \rightsquigarrow -\frac{S(\xi(\cdot))}{f(\xi(\cdot))} \int_{(0,\xi(\cdot)]} \frac{d\mathbb{Z}}{(1 - \Delta\Lambda)}$$