Preliminaries for Semiparametric Inference

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Outline

Chapter 17

- 17.0 Introduction
- 17.1 Projections
- 17.2 Hilbert Spaces
- 17.3 More on Banach Spaces

Introduction

In the following slides, we will follow chapter 17 in presenting the technical background needed for the development of semiparametric inference theory.

We will begin first with a general treatment of projections, followed by thorough consideration of Hilbert spaces. Finally, we will revisit the Banach space as covered in Chapter 6.

In all, our intention is to provide the background needed for a "semiparametric efficiency calculus", the development of which will be the focus of the first few chapters in this unit.

17.1 Projections

The projection of an object T onto a space $\mathcal{S}$ is the element $\hat{S} \in \mathcal{S}$ which is "closest" to T in some sense.

In the semiparametric case, the object T is usually a random variable, and the spaces of interest contain square-integrable random variables.

Theorem 17.1 gives us a simple method for identifying the projection of T in this setting.

Theorem 17.1

Let $\mathcal{S}$ be a linear space of real random variables with finite second moments. Then $\hat{S}$ is the projection of T onto $\mathcal{S}$ if and only if:

(i) $\hat{S} \in \mathcal{S}$

(ii) $E(T - \hat{S})S = 0$ for all $S \in \mathcal{S}$.

If S_1 and S_2 are both projections, then $S_1 = S_2$ almost surely. If $\mathcal{S}$ contains the constants, then $ET = E\hat{S}$ and $cov(T - \hat{S}, S) = 0$ for all $S \in \mathcal{S}$.

Theorem 17,1 Proof

Proof: $\Longrightarrow$
Begin by assuming that conditions (i) and (ii) hold. Then for any $S \in \mathcal{S}$, we have:

$$E(T-S)^2 = E(T-\hat{S})^2 + 2E(T-\hat{S})(\hat{S}-S) + E(\hat{S}-S)^2 \quad (17.1)$$

However, condition (ii) forces the middle term to be zero, such that $E(T-S)^2 \geq E(T-\hat{S})^2$, with strict inequality whenever $E(\hat{S}-S)^2 > 0$.

Thus, $\hat{S}$ is almost surely unique projection of T onto $\mathcal{S}$.

Proof: $\Longleftarrow$
Assume that $\hat{S}$ is a projection, and note that for any $\alpha \in \mathbb{R}$ and any $S \in \mathcal{S}$,

$$E(T - \hat{S} - \alpha S)^2 - E(T - \hat{S})^2 = -2\alpha E(T - \hat{S})S + \alpha^2 ES^2$$

Since $\hat{S}$ is a projection, the left side is strictly nonnegative for every α.

However, considering the parabola $\alpha \mapsto \alpha^2 ES^2 - 2\alpha E(T - \hat{S})S$, we can note that this parabola is nonnegative for all α and S if and only if $E(T - \hat{S})S = 0$ for all S.

Finally, uniqueness of the projection $\hat{S}$ follows from considering two candidate projections S_1 and S_2 and applying equation (17.1):

$$E(T - S)^2 = E(T - \hat{S})^2 + 2E(T - \hat{S})(\hat{S} - S) + E(\hat{S} - S)^2$$

to both S_1 and S_2, forcing $E(S_1 - S_2)^2 = 0$, thus any such projection is unique.

If the constants are in $\mathcal{S}$, then theorem 17.1 gives that $E(T - \hat{S})c = 0$ for any $c \in \mathbb{R}$. Taking $c = 1$ gives us the remaining assertions.

Note that theorem 17.1 does not provide that a projection always exist. One can take the example that $\mathcal{S}$ is open in the $L_2(P)$ norm, then the infimum of $E(T-S)^2$ over $S \in \mathcal{S}$ is not achieved.

A sufficient condition for existence follows directly: that $\mathcal{S}$ is closed in the $L_2(P)$ norm. However, we can often establish the existence of a projection directly.

Such existence questions will be handled more in the upcoming chapters in this unit.

Projection Example

A very useful example of a projection comes from conditional expectation. Let X and Y be real random variables on a probability space. Then $g_0(y) \equiv E(X|Y = y)$ is the conditional expectation of X given $Y = y$.

If we let $\mathcal{G}$ be the space of all measurable functions g of Y such that $Eg^2Y < \infty$, then by verifying that:

$$E(X - g_0(Y))g(Y) = 0$$

for all $g \in \mathcal{G}$, we have (provided $Eg_0^2(Y) < \infty$), $E(X|Y = y)$ is the projection of X onto the space $\mathcal{G}$. Theorem 17.1 guarantees that the conditional expectation is almost surely unique.

We begin our discussion of Hilbert spaces, which are essentially generalizations of finite-dimensional Euclidean spaces. Additionally, they serve as a specific case of the Banach space, and like Banach spaces, are often infinite-dimensional.

Precisely, a Hilbert space is a Banach space which is equipped with an inner product. An inner product on a Banach space $\mathbb{D}$ with norm $\|\cdot\|$, is a function $\langle\cdot,\cdot\rangle : \mathbb{D} \times \mathbb{D} \mapsto \mathbb{R}$., such that for all $\alpha, \beta \in \mathbb{R}$ and $x, y, z \in \mathbb{D}$:

(i) $\langle x, x\rangle = \|x\|^2$
(ii) $\langle x, y\rangle = \langle y, x\rangle$
(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

We can also define the semi-inner product when $\|\cdot\|$ is a semi-norm.

It is also possible to begin with an inner product, and then generate a norm rather than vice-versa.

Begin with a linear space $\mathbb{D}$ with a semi-inner product $\langle \cdot, \cdot \rangle$ which satisfies the "linear in first argument" symmetry properties stated above, and also satisfies $\langle \cdot, \cdot \rangle \geq 0$.

With this, taking $\|x\| = \langle x, x \rangle^{1/2}$ for $x \in \mathbb{D}$ defines a semi-norm on $\mathbb{D}$. Theorem 17.2 verifies this.

Theorem 17.2

Let $\langle\cdot,\cdot\rangle$ be a semi-inner product on $\mathbb{D}$, with $\|x\| = \langle x, x\rangle^{1/2}$ for all $x \in \mathbb{D}$. Then for all $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{D}$:

(a) $\langle x, y\rangle \leq \|x\|\|y\|$
(b) $\|x + y\| \leq \|x\| + \|y\|$
(c) $\|\alpha x\| = |\alpha| \times \|x\|$

$\langle\cdot,\cdot\rangle$ is also an inner product, then $\|x\| = 0$ if and only if $x = 0$

A proof is provided on the following slides.

Proof

Note that:

$$0 \leq \langle x - \alpha y, x - \alpha y \rangle = \langle x, x \rangle - 2\alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle$$

allowing $a \equiv \langle y, y \rangle$, $b \equiv -2|\langle x, y \rangle|$, $c \equiv \langle x, x \rangle$, and $t \equiv \alpha \mathrm{sign} \langle x, y \rangle$, we have:

$$q(t) \equiv \leq at^2 + bt + c \geq 0$$

which is a quadratic equation in t forced to be greater than 0. Thus, $q(t)$ has at most one real root, implying the discriminant of the quadratic formula is not positive:

$$0 \geq b^2 - 4ac = 4\langle x, y \rangle^2 - 4\langle x, x \rangle \langle y, y \rangle$$

solving yields (a), that $\langle x, y \rangle \leq \|x\| \|y\|$.

Proof

Now that we've shown (a), (b) follows directly since:

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2\end{aligned}$$

finally, (c) follows from the equalities:

$$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\langle \alpha x, x\rangle = \alpha^2\langle x, y\rangle = (\alpha\|x\|)^2$$

Part (a) in Theorem 17.2 is the Cauchy-Schwartz inequality.

We will refer to two elements, x, y, in a Hilbert space $\mathbb{H}$, as *orthogonal* if $\langle x, y \rangle = 0$, denoted as $x \perp y$.

For any set $C \subset \mathbb{H}$ and any $x \in \mathbb{H}$, x is orthogonal to C if $x \perp y$ for all $y \in C$, denoted $x \perp C$. Two subsets $C_1, C_2 \subset \mathbb{H}$ are orthogonal, denoted $C_1 \perp C_2$ if all of their elements are orthogonal.

For any set $C_1 \subset \mathbb{H}$, the *orthocomplement* of C_1 denoted $C_1^{\perp}$ is the set $\{x \in \mathbb{H} : x \perp C_1\}$

Let the subspace $H \subset \mathbb{H}$ be linear and closed. Theorem 17.1, our core projection result, provides that for any $x \in \mathbb{H}$, there exists an element $y \in H$ that satisfies $\|x - y\| \leq \|x - z\|$, and such that $\langle x - y, z \rangle = 0$ for all $z \in H$.

Let Π be an operator which performs this action, such that $\Pi x \equiv y$ for the above example. This "projection" operator, $\Pi : \mathbb{H} \mapsto H$ has several important properties, defined in Theorem 17.3.

First, recall the definitions, for an operator T, of the null space $N(T)$ and range $R(T)$:

$$N(T) \equiv \{x \in \mathbb{H} : Tx = 0\}$$

$$R(T) \equiv \{y \in \mathbb{H} : Tx = y \text{ for some } x \in \mathbb{H}\}$$

Theorem 17.3

Let H be a closed, linear subspace of $\mathbb{H}$ and let $\Pi : \mathbb{H} \mapsto H$ be the projection operator onto H. Then:

(i) Π is continuous and linear,
(ii) $\|\Pi x\| \leq \|x\|$ for all $x \in \mathbb{H}$
(iii) $\Pi^2 \equiv \Pi\Pi = \Pi$, and:
(iv) $N(\Pi) = H^{\perp}$ and $R(\Pi) = H$.

We present the proof for theorem 17.3 on the following slides.

Proof Theorem 17.3

Proof

Let $x, y \in \mathbb{H}$ and $\alpha, \beta \in \mathbb{R}$ If $z \in H$ then:

$$\langle [\alpha x + \beta y] - [\alpha \Pi x + \beta \Pi y], z \rangle = \alpha \langle x - \Pi x, z \rangle + \beta \langle y - \Pi y, z \rangle = 0$$

By the theorem 17.1, we now have that:
$\Pi(\alpha x + \beta y) = \alpha \Pi x + \beta \Pi y$. This provides the linearity of Π.
Proving (ii) will provide continuity required for (i).

Since $\langle x - \Pi x, \Pi x \rangle = 0$ for any $x \in \mathbb{H}$, we have that:

$$\|x\|^2 = \|x - \Pi x\|^2 + \|\Pi x\|^2 \geq \|\Pi x\|^2$$

This yields (ii), and additionally (i).

Proof
For (iii), consider any $y \in H$. It's trivial that $\Pi y = y$. Thus, for any $x \in \mathbb{H}, \Pi(\Pi x) = \Pi(y) = y = \Pi x$.

Finally, for $x \in N(\Pi)$, we have $x = x - \Pi x \in H^{\perp}$. Thus $N(\Pi) \subset H^{\perp}$. Additionally for $x \in H^{\perp}, \Pi x = 0$ by definition, thus implying $H^{\perp} \subset N(\Pi)$.

Finally, $R(\Pi) \subset H$ follows from its definition. For any $x \in H$, $\Pi x = x$, and thus $H \subset R(\Pi)$. This completes (iv), and thus the proof.

For any projection Π onto a closed linear subspace $H \in \mathbb{H}$, $\mathrm{I} - \Pi$, where I is the identity operator, is a projection onto the closed linear subspace $H^{\perp}$.

A noteworthy example of a Hilbert space is $\mathbb{H} = L_2(P)$ with the inner product $\langle f, g \rangle = \int fgdP$. A closed subspace of interest is $L_2^0(P) \subset L_2(P)$, consisting of all mean zero functions in $L_2(P)$.

The projection operator $\Pi : L_2(P) \mapsto L_2^0(P)$ is $\Pi x = x - Px$. In order to see this, note that $\langle x - \Pi x, y \rangle = \langle Px, y \rangle = PxPy = 0$ for any $y \in L_2(P)^0$.

Consider the situation where we have two closed, linear subspaces $H_1, H_2 \subset \mathbb{H}$ where H_1 and H_2 are not necessarily orthogonal. Let Π_j be the projection onto H_j, and define $Q_j = I - \Pi_j$ for $j = 1, 2$.

The *sumspace* of H_1 and H_2 is $H_1 + H_2 \equiv \{h_1 + h_2 : h_1 \in H_1, h_2 \in H_2\}$.

We consider the idea of *alternating projections*, in which we alternate between projection of some $h \in \mathbb{H}$ onto the orthocomplements of H_1 and H_2 repeatedly, so that in the limit we obtain the projection, $\tilde{h}$, of h onto the orthocomplement of the closure of the sumspace of H_1 and H_2.

Our intention is that $h - \tilde{h}$ is the projection of h onto $\overline{H_1 + H_2}$.

Let $h_j^{(m)} = \Pi_j \left[\mathrm{I} - (Q_1 Q_2)^m \right] h$ and $\tilde{h}_j^{(m)} = \Pi_j \left[\mathrm{I} - (Q_2 Q_1)^m \right] h$. Let Π project onto $\overline{H_1 + H_2}$ and $Q \equiv \mathrm{I} - \Pi$.

Theorem 17.4 provides that Π in this setting can be computed as the limit of iterations between Q_2 and Q_1, and that Πh can be expressed as a sum of elements in H_1 and H_2, given certain conditions.

We will state theorem 17.4 without proof following the textbook.

Theorem 17.4

Assume $H_1, H_2 \subset H$ are closed and linear. Then, for any $h \in \mathbb{H}$:

(i) $\|h_1^{(m)} + h_2^{(m)} - \Pi h\| \equiv u_m \to 0$ as $m \to \infty$
(ii) $\|\tilde{h}_1^{(m)} + \tilde{h}_2^{(m)} - \Pi h\| \equiv \tilde{u}_m \to 0$ as $m \to \infty$
(iii) $u_m \vee \tilde{u}_m \leq \rho^{2(m-1)}$, where ρ is the cosine of the minimum angle τ between H_1 and H_2 considering only elements in $(H_1 \cap H_2)^{\perp}$.
(iv) $\rho < 1$ if and only if $H_1 + H_2$ is closed; and
(v) $\|I - (Q_2 Q_1)^m - \Pi\| = \|I - (Q_1 Q_2)^m - \Pi\| = \rho^{2m-1}$

We now focus our attention to *linear functionals* on Hilbert spaces.

Recall that a linear operator is an operator for which $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, and that the norm for a linear operator $T : \mathbb{D} \mapsto \mathbb{E}$ is $\|T\| \equiv \sup_{x \in \mathbb{D}: \|x\| \leq 1} \|Tx\|$

In the special case when the output space $\mathbb{E} = \mathbb{R}$, a linear operator is called a linear functional. We call a linear functional bounded when $\|T\| < \infty$. In this setting, boundedness of a linear operator is equivalent to continuity (see proposition 6.15).

Theorem 7.15 gives a very important result for bounded linear functionals in Hilbert spaces.

Theorem 17.5 (Riesz representation theorem)

If $L : \mathbb{H} \mapsto \mathbb{R}$ is a bounded linear functional on a Hilbert space, then there exists a unique element $h_0 \in \mathbb{H}$ such that $L(h) = \langle h, h_0 \rangle$ for all $h \in \mathbb{H}$ and moreover, $\|L\| = \|h_0\|$

Proof

Assume that L exists and let $H = N(L)$. Note that H is closed and linear because L is continuous and the space $\{0\}$ is trivially closed and linear. Assume that $H \neq \mathbb{H}$ as otherwise the proof would be trivial with $h_0 = 0$.

Proof

Thus, there exists an object $f_0 \in H^\perp$ such that $L(f_0) = 1$. Hence, for all $h \in \mathbb{H}$, $h - L(h)f_0 \in H$ since:

$$L(h - L(h)f_0) = L(h) - L(h)L(f_0) = 0$$

Since H and $H^\perp$ are orthogonal, we have for all $h \in \mathbb{H}$:

$$0 = \langle h - L(h)f_0, f_0 \rangle = \langle h, f_0 \rangle - L(h)\|f_0\|^2$$

Setting $h_0 \equiv \|f_0\|^{-2} f_0$, $L(h) = \langle h, h_0 \rangle$ for all $h \in \mathbb{H}$.

Suppose $h_0^{'} \in \mathbb{H}$ satisfies $\langle h, h_0^{'} \rangle = \langle h, h_0 \rangle$ for all $h \in \mathbb{H}$, then $(h_0 - h_0^{'}) \perp \mathbb{H}$, thus $h_0 = h_0^{'}$. Since by the Cauchy-Schwartz inequality, $|\langle h, h_0 \rangle| \leq \|h\|\|h_0\|$ and $\langle h_0, h_0 \rangle = \|h_0\|^2$, we have that $\|L\| = \|h_0\|$, as required.

Recall the definition of a Banach space as a complete normed space. Similar to Hilbert spaces, a linear functional on a Banach space is just a linear operator with real range.

The dual space $\mathbb{B}^*$ of a Banach space $\mathbb{B}$ is the set of all continuous linear functions on $\mathbb{B}$.

Application of proposition 6.15 yields readily that for every $b^* \in \mathbb{B}^*$, we have: $|b^*b| \leq \|b^*\|\|b\|$ for every $b \in \mathbb{B}$, where $\|b^*\| \equiv \sup_{b\in\mathbb{B}:\|b\|\leq 1}|b^*b| \leq \infty$.

For a Hilbert space $\mathbb{H}$, $\mathbb{H}^*$ can be identified with $\mathbb{H}$ by the Reisz representation theorem given above.

This implies the existence of an *isometry* between $\mathbb{H}$ and $\mathbb{H}^*$. Recall that an isometry is a one-to-one correspondence between spaces which preserves norms.

To see this, select an arbitrary $h^* \in \mathbb{H}$, and let $\tilde{h} \in \mathbb{H}$ be the unique element that satisifies $\langle h, \tilde{h} \rangle = h^* h$ for all $h \in \mathbb{H}$. Then:

$$\|h^*\| = \sup_{h \in \mathbb{H} : \|h\| \leq 1} \langle h, \tilde{h} \rangle| \leq \|\tilde{h}\|$$

which follows from the Cauchy-Schwartz inequality. Since h^* is arbitrary, the isometry conclusion follows directly.

Considering two Banach spaces, for each continuous operator between them $A : \mathbb{B}_1 \mapsto \mathbb{B}_2$, there exists an *adjoint map*, or adjoint, $A^* : \mathbb{B}_2^* \mapsto \mathbb{B}_1^*$ defined as $(A^* b_2^*) b_1 = b_2^* A b_1$ for all $b_1 \in \mathbb{B}_1$ and $b_2^* \in \mathbb{B}_2^*$.

It follows directly that A^* is linear. Proposition 17.6, presented on the following slide, presents that A^* is bounded and thus continuous.

Proposition 17.6

Let $A : \mathbb{B}_1 \mapsto \mathbb{B}_2$ be a bounded linear operator between Banach spaces. Then $\|A^*\| = \|A\|$.

Proof Given any $b_2^* \in \mathbb{B}_2^*$:

$$\begin{aligned}\|A^* b_2^*\| &= \sup_{b_1 \in \mathbb{B}_1 : \|b_1\| \leq 1} |A^* b_2^* b_1| \\ &= \sup_{b_1 \in \mathbb{B}_1 : \|b_1\| \leq 1} \left\{ \left| b_2^* \left(\frac{Ab_1}{\|Ab_1\|} \right) \right| \|Ab_1\| \right\} \\ &\leq \|b_2^*\| \|A\|\end{aligned}$$

we have $\|A^*\| \leq \|A\|$. Thus $\|A^*\|$ is a continuous, linear operator.

Proof of Proposition 17.6

Now let $A^{**} : \mathbb{B}_1^{**} \mapsto \mathbb{B}_2^{**}$ be the adjoint of A^* with respect to the dual of the duals of $\mathbb{B}_1$ and $\mathbb{B}_2$. Note that for any $j = 1, 2$, the map $b_j : \mathbb{B}_j \mapsto \mathbb{R}$ defined by $b_j^* \mapsto b_j^* b_j$ is a bounded linear function, yielding $\mathbb{B}_j \subset \mathbb{B}_j^{**}$.

By these definitions, for any $b_1 \in \mathbb{B}_1$ and $b_2^* \in \mathbb{B}_2^*$:

$$(A^{**} b_1) b_2^* = (A^* b_2^*) b_1 = b_2^* A b_1$$

which yields that $\|A^{**}\| \leq \|A^*\|$, and the restriction of A^{**} to $\mathbb{B}_1$ (call this A_1^{**}), is equal to A. Hence $\|A\| = \|A_1^{**}\| \leq \|A^*\|$, yielding the desired result.

For two Hilbert spaces $A : \mathbb{H}_1 \mapsto \mathbb{H}_2$ between two Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ the adjoint is a map $A^* : \mathbb{H}_2 \mapsto \mathbb{H}_1$ satisfying $\langle Ah_1, h_2 \rangle_2 = \langle h_1, A^* h_2 \rangle_1$ for every $h_1 \in \mathbb{H}_1$ and $h_2 \in \mathbb{H}_2$.

Considering the adjoint of a restriction of a continuous linear operator $A : \mathbb{H}_1 \mapsto \mathbb{H}_2$, $A_0 : \mathbb{H}_{0,1} \mapsto \mathbb{H}_2$ where $\mathbb{H}_{0,1}$ is a closed linear subspace of $\mathbb{H}_1$. If $\Pi : \mathbb{H}_1 \mapsto \mathbb{H}_{0,1}$ is the projection onto the subspace, the adjoint of A_0 is $A_0^* = \Pi \circ A^*$

Denote $B(\mathbb{D}, \mathbb{E})$ as the collection of all linear operators between normed spaces $\mathbb{D}$ and $\mathbb{E}$.

Lemma 6.16 yields that for any $T \in B(\mathbb{B}_1, \mathbb{B}_2)$, $R(T)$ is not closed unless T is continuously invertible.

Consider this counter-example: let $\mathbb{B}_1 = \mathbb{B}_2 = L_2(0,1)$, and define $T : L_2(0,1) \mapsto L_2(0,1)$ by $Tx(u) = ux(u)$. It follows that $\|T\| \leq 1$, and thus $T \in B(L_2(0,1), L_2(0,1))$. The range of T is:

$$R(T) = \left\{ y \in L_2(0,1) : \int_0^1 u^{-2} y^2(u) du < \infty \right\}$$

however, the functions $y_1(u) = 1$ and $y_2(u) = \sqrt{u}$ are not contained in $R(T)$, and thus $R(T)$ is not closed.

$R(T)$'s lack of closure is due to the inverse operator $T^{-1}y(u) = u^{-1}y(u)$ is not bounded over $L_2(0,1)$. However, one can verify that for any normed spaces $\mathbb{D}$ and $\mathbb{E}$ and any $T \in B(\mathbb{D}, \mathbb{E}), N(T)$ is always closed due to the continuity of T. Observe that also:

$$\begin{aligned} N(T^*) &= \{b_2^* \in \mathbb{B}_2^* : (T^* b_2^*) b_1 = 0 \text{ for all } b_1 \in \mathbb{B}_1\} \\ &= \{b_2^* \in \mathbb{B}_2^* : b_2^*(T b_1) = 0 \text{ for all } b_1 \in \mathbb{B}_1\} \\ &= R(T)^\perp \end{aligned}$$

where $R(T)^\perp$ is the set of linear functionals in $\mathbb{B}_2^*$ that yield zero on $R(T)$. Theorem 17.7, proven on the following slides, extends the relationship above. We begin by stating a necessary lemma, 7.18, without proof.

Theorem 17.7

For two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$ and for any $T \in B(\mathbb{B}_1, \mathbb{B}_2)$, $R(T) = \mathbb{B}_2$ if and only if $N(T^*) = \{0\}$ and $R(T^*)$ is closed.

Proof

If $R(T) = \mathbb{B}_2$, then $0 = R(T)^{\perp} = N(T^*)$, and thus T^* is one-to-one. Since $\mathbb{B}_2$ is closed, it must be that $R(T^*)$ is closed (which follows from Lemma 17.8). For the converse, if $N(T^*) = \{0\}$ and $R(T^*)$ is closed, then $R(T)^{\perp} = \{0\}$ and $R(T)$ is closed, which $R(T) = \mathbb{B}_2$

Lemma 17.8

Theorem 17.9

For two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, and for any $T \in B(\mathbb{B}_1, \mathbb{B}_2)$, $R(T)$ is closed if and only if $R(T^*)$ is closed.

This result is familiar one from real analysis, similar to the alternative definitions of functional continuity.

Theorem 17.7

For two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$ and for any $T \in B(\mathbb{B}_1, \mathbb{B}_2)$, $R(T) = \mathbb{B}_2$ if and only if $N(T^*) = \{0\}$ and $R(T^*)$ is closed.

Proof

If $R(T) = \mathbb{B}_2$, then $0 = R(T)^\perp = N(T^*)$, and thus T^* is one-to-one. Since $\mathbb{B}_2$ is closed, it must be that $R(T^*)$ is closed (which follows from Lemma 17.8). For the converse, if $N(T^*) = \{0\}$ and $R(T^*)$ is closed, then $R(T)^\perp = \{0\}$ and $R(T)$ is closed, which $R(T) = \mathbb{B}_2$

Restricting Theorem 17.7 to Hilbert spaces, we obtain trivially that for any $A \in B(\mathbb{H}_1, \mathbb{H}_2)$ that $R(A)^{\perp} = N(A^*)$.

The final result of the Chapter, theorem 17.8, gives a useful result for Hilbert spaces.

Theorem 17.9

For two Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, and for any $A \in B(\mathbb{H}_1, \mathbb{H}_2)$, $R(A)$ is closed if and only if $R(A^*)$ is closed if and only if $R(A^*A)$ is closed. Moreover, if $R(A)$ is closed then $R(A^*) = R(A^*A)$ and:

$$A(A^*A)^{-1}A^* : \mathbb{H}_2 \mapsto \mathbb{H}_1$$

is the projection onto $R(A)$

Proof

The first result holds easily by lemma 17.8, but following the text, will prove this again for the specialization to Hilbert spaces ot highlight interesting features that will become useful later.

Proof

First assume $R(A)$ is closed, and let A_0^* be the restriction of A^* to $R(A)$, and note that $R(A^*) = R(A_0^*)$, since $R(A)^\perp = N(A^*)$.

Let A_0 be the restriction of A to $N(A)^\perp$, and note that $R(A_0) = R(A)$ by definition of $N(A)$. It follows that $R(A_0)$ is closed, and $N(A_0) = \{0\}$.

Lemma 6.16 guarantees the existence of a constant $c > 0$ such that $\|A_0 x\| \geq c\|x\|$ for all $x \in N(A)^\perp$

Proof

Now, pick a $y \in R(A_0)$ and note that there exists an $x \in N(A)^\perp$ such that $y = A_0 x$, yielding:

$$\|x\| \|A_0^* y\| \geq \langle x, A_0^* y \rangle = \langle A_0^* x, y \rangle = \|A_0 x\| \|y\| \geq c \|x\| \|y\|,$$

and therefore, $\|A_0^* y\| \geq \|y\|$ for all $y \in R(A_0)$. This yields that $R(A^*) = R(A_0^*)$ is closed.

Proof of Theorem 17.9

Proof

Now assume $R(A^*)$ is closed. Using the same argument as the previous slide, $R(A^{**})$ is coed. However, due to the restriction to Hilbert spaces, $A^{**} - A$, and thus $R(A)$ is closed.

Now assume that either $R(A)$ or $R(A^*)$ is closed, we know from our previous argument that both must be closed. It follows again from the argument above that $R(A^*A) = R(A_0^*A_0)$.

Lemma 6.16 again gives the existence of $c_1, c_2 > 0$ such that $\|A_0 x\| \geq c_1 \|x\|$ and $\|A_0^* y\| \geq c_2 \|y\|$ for all $x \in N(A)^\perp$ and $y \in R(A)$

Proof

Thus for all $x \in N(A_0)^{\perp}$ we have:

$$\|A_0^* A_0 x\| \geq c_2 \|A_0 x\| \geq c_1 c_2 \|x\|$$

yielding that $R(A^* A) = R(A_0^* A_0^*)$ is closed.

Now that $R(A^* A)$ is closed. Thus $R(A_0^* A_0^*)$ is closed, and by recycling arguments, we have that: $N(A_0^* A_0) = N(A_0) = \{0\}$. Thus there exists a $c > 0$ such that all $x \in N(A_0)^{\perp}$,

$$c\|x\| \leq \|A_0^* A_0 x\| \leq \|A_0^*\| \|A_0 x\|$$

and therefore, $R(A) = R(A_0)$ is closed, and thus $R(A^*)$ and $R(A^* A)$ are also closed.

Proof of Theorem 17.9

Proof

Note that $R(A^*A)$ is clearly a subset of $R(A^*)$. Since $R(A)^\perp = N(A^*)$, we also have that $R(A^*) \subset R(A^*A)$, and thus $R(A^*) = R(A^*A)$.

This yields, by lemma 6.16, that A^*A is continuously invertible on $R(A^*)$, and thus $A(A^*A)^{-1}A^*$ exists and is well defined.

We then have that for any $y \in R(A)$, there exists an $x \in \mathbb{H}_1$ such that $y = Ax$. Thus:

$$\Pi y \equiv A(A^*A)^{-1}A^*y = A(A^*A)^{-1}(A^*A)x = Ax = y$$

for any $y \in R(A)^\perp$, we have that $y \in N(A^*)$ and thus $\Pi y = 0$. Hence Π is the projection operator onto $R(A)$, by definition, and the proof is complete.